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On Vague Objects, Fuzzy Logic and Fractal Boundaries

B.J. Copeland

I

Are there vague objects? Semantic indeterminacy analyses enable us to answer no if we wish; and from a technical point of view this answer undoubtedly saves a lot of bother. Yet I find it hard to disagree with the proposition that there are such objects. I take the Australian outback to be an example of one. David Lewis denies that 'there's this thing, the outback, with imprecise borders' (1986, p.212). But this view seems to me to go against common sense. There is such an object. I've driven across it. And it simply is a fact about the outback that it has a fuzzy boundary.

If there are vague objects then it seems to me that there must be indeterminate identity statements. In the case where 'a' names a vague object and 'b' names a precise object then one may plausibly maintain that the statement a=b, while vague, is false. The same can be said in certain cases where both 'a' and 'b' name vague objects (the outback and the steppes, for example) but not, I think, in all such cases.[1] Let 'New Devon' be the name given by a 17th century British seafarer to a natural harbour and indeterminate amount of hinterland on the coastline of what is now Western Australia. Those present at the naming did not know that a year previously, at a similar ceremony on roughly the same spot, a French mariner had named the district 'Nouvelle Provence'. The statement 'New Devon = Nouvelle Provence' is not false but indeterminate, or so it seems to me.

These facts, if facts they be, are of no help in determining whether vagueness comes into the world only with the use of representations or whether, in some deep sense, the world *in itself* is vague. The outback and the maritime region of New Devon are certainly objects in the world but there is surely nothing in these anodyne examples to make us suspect that vagueness might be other than the product of the way we think and speak.

This notion that the world in itself might be vague strikes me as a particularly dark piece of metaphysics. Mark Sainsbury searches tolerantly for a way of giving coherent expression to the notion and his conclusion is that 'when we make enough concessive assumptions to have an intelligible thesis of ontic vagueness, we have a thesis which can be shown to be false by a few short lines of proof' (1994, p.7). As I will explain I don't think Sainsbury has managed to find a way of giving coherent expression to the thesis of ontic vagueness (or in other words to the thesis that the world in itself is vague). Nor do I think the few short lines of proof that Sainsbury is referring to (Evans 1978) show very much at all.

II

Evans reads his delta operators $\triangle$ and ∇ 'Definitely' and 'Indefinitely' respectively. I prefer 'It is determinate whether' and 'It is indeterminate whether', since 'Definitely A' is too easily confused with the quite different statement 'A is definitely true'. (Evans remarked in correspondence with Lewis that he did not intend $\triangle A$ to imply A.[2]) His famous proof is as follows.

(1) $\nabla(a=b)$

(2) $\lambda x[\nabla(x=a)]b$ from (1)

(3) $\neg\nabla(a=a)$ axiom

(4) $\neg\lambda x[\nabla(x=a)]a$ from (3)

(5) $a\neq b$ from (2) and (4) by Leibniz's Law

How does this proof fare if, accepting Evans' invitation to take indeterminateness seriously, we consider an infinity of statement-values? Let every statement under consideration have a value lying in the closed interval [0, 1] of the real line. This is the framework used in *fuzzy* logic. (Fuzzy logic aims to extend deductive methods to situations in which the information available may be only partly or approximately true.[3]) I will call 0 and 1 the integral values. The result of prefixing a statement with a delta operator is always a statement with an integral value. In particular:

$v(\Delta A) = 1$ iff either $v(A) = 1$ or $v(A) = 0$

$v(\nabla A) = 1$ iff $0 < v(A) < 1$.

Also (Zadeh 1975):

$v(\neg A) = 1 - v(A)$

$v(A\&B) = \min(v(A), v(B))$

$v(A\rightarrow B) = 1$ if $v(B) \geq v(A)$, $1 - (v(A) - v(B))$ otherwise.[4]

Notice that the delta operators are duals, as Evans requires.

Prima facie Evans' proof serves only to remind us of the obvious, namely that if $\nabla(a=b)$ is true then $a\neq b$ is partly true. Let me refer to $\nabla(a=b)\rightarrow a\neq b$ as 'Evans' conditional'. When the value of $a=b$ is integral this conditional takes the value 1. Where the value of $a=b$ is non-integral the conditional never takes the value 0 although the higher the value of $a=b$ the lower the value of the conditional. If the value of $a=b$ is non-integral the value of the antecedent is always 1 but when the value of the conditional is low detaching $a\neq b$ is perilous. If the value of the conditional is high modus ponens may be applied but it must be remembered that in fuzzy logic the safety of an inference is a matter of degree and $a\neq b$, once detached, must not be confused with, nor taken to imply, $a\neq b\&\Delta a\neq b$. In a framework where truth is a matter of degree, then, there is no inconsistency involved in lines (1) and (5) of the proof both being true to a high degree (1 and .99 respectively, for example); and in favourable circumstances the statement that line (1) of the proof implies line (5) is itself true to a high degree.

So how is Evans' proof supposed to challenge anything, let alone 'the idea that the world might contain certain objects about which it is a fact that they have fuzzy boundaries' (Evans, op. cit.)? Evans' first attempt to generate a challenge is to maintain that the final line of his proof, $a\neq b$, 'contradict[s] the assumption, with which we began, that the identity statement $a=b$ is of indeterminate truth value' (ibid.). That is to say, Evans is here assuming

$$\neg(a\neq b\&\nabla(a=b));$$

or equivalently

$$\nabla(a=b)\rightarrow a=b;$$

or equivalently again (by contraposition, duality and double negation)

$$a\neq b\rightarrow\Delta(a\neq b).$$

Let me refer to this latter conditional as 'Evans' axiom'.

If one is thinking classically then Evans' axiom and Evans' conditional do jointly entail his desired conclusion $\neg\nabla(a=b)$. But it is precisely such thinking as this that the presence of the delta operators should discourage. Only when the value of a=b is integral do Evans' axiom and Evans' conditional both take the value 1. For non-integral values of a=b these two conditionals form an *opposed pair*. That is to say, driving the value of either of them up drives the value of the other down, and vice versa. Figure 1 plots the value of $\nabla(a=b)\rightarrow a\neq b$ against the value of $a\neq b\rightarrow\Delta(a\neq b)$ for non-integral values of a=b.

FIGURE 1 ABOUT HERE

The best compromise attainable is at the midpoint, where both conditionals take the value .5. But no one should want to base their rejection of a philosophical position on a pair of half-truths.

Even without recourse to the formal semantics sketched above it is easy to question the conditional that I'm calling Evans' axiom. For as I have remarked it is equivalent to $\nabla(a=b)\rightarrow a=b$. Why should we think it true that if it is indeterminate whether a=b, then a=b? Yet if $\nabla(a=b)\rightarrow a=b$ is not true then nor is Evans' seemingly crucial claim that a≠b 'contradict[s] the assumption ... that the identity statement a=b is of indeterminate truth value'.

Friends of Evans' argument may retort that I am reading him too literally. Evans certainly *says* that $\nabla(a=b)$ and a≠b contradict, but perhaps this is not quite what he meant. What then did he mean? One thing that he might have meant can be brought out by considering the rule of proof construction known as necessitation. Once P is established *in the course of a proof* then $\Box$P may be inferred as a subsequent line. In general, of course, P does not imply $\Box P$, and $\Diamond P$ and $\neg P$ do not contradict; but if $\Diamond P$ and $\neg P$ are both obtained in the course of a proof then contradiction is just a step away. Perhaps Evans' intention, then, was not to endorse $a\neq b\rightarrow\Delta(a\neq b)$ but rather to endorse a rule of proof that may be termed Δ-introduction (ΔI): once P is established in the course of a proof, ΔP may be inferred. An application of (ΔI) then takes us from line (5) to

(5') $\Delta(a\neq b)$

which, given duality, does contradict line (1). This interpretation of what Evans was about was suggested to me by Peter Smith (in correspondence).

(ΔI) is certainly admissible in contexts in which any statement that is inferred or assumed in the course of a proof is determinately true. The difficulty is that in the present setting the inference from (5) to (5') is invalid, since the derived formula has a lower value than the formula from which it is derived (for under the assumption that $v(\nabla(a=b))=1$, $v(a\neq b)$ is non-integral and so $v(\Delta(a\neq b))=0$.) To embrace either $a\neq b\rightarrow\Delta(a\neq b)$ as an axiom or (ΔI) as a rule is to fail to take indeterminacy seriously.

Evans' second attempt to turn his derivation into a reductio involves the suggestion that 'Leibniz's Law ... be strengthened with a "Definitely" prefix' (ibid.).[5] (I assume that Evans takes Leibniz's Law to be the schema $a=b\rightarrow(\Phi(a)\leftrightarrow\Phi(b))$.) In the present context this proposal is indefensible. As the value of $a=b$ decreases so may the value of $\Phi(a)\leftrightarrow\Phi(b)$. This is exactly what we should expect: if $a=b$ is true only to a degree then it can hardly be determinately the case that a and b have all their properties in common. If any decrease in the value of the antecedent of Leibniz's Law is accompanied by an identical decrease in the value of the consequent then the Law retains the value 1, as it does if the value of $a=b$ decreases more swiftly than the value of $\Phi(a)\leftrightarrow\Phi(b)$ or if the value of the latter remains constant. The difficulty for Evans is that where Φ contains occurrences of delta operators, the value of $\Phi(a)\leftrightarrow\Phi(b)$ may become 0 very early in the descent of the value of $a=b$, in which case the prefixed form of Leibniz's Law takes the value 0. In particular, where $v(\lambda x[\nabla(x=a)]b)$ takes the value 1, the value of the conditional

$$a=b\rightarrow(\lambda x[\nabla(x=a)]a\leftrightarrow\lambda x[\nabla(x=a)]b)$$

is non-integral. It is, of course, this latter fact that explains why the final inference of Evans' proof yields a statement of non-integral value.

Let me say that A determinately implies B just in case $v(A\rightarrow B)=1$. An advocate of Evans' proof can sweep away all difficulties by declaring $a=b$ to determinately imply $\lambda x[\nabla(x=a)]a\leftrightarrow\lambda x[\nabla(x=a)]b$, for this blocks the claim that the value of (5) is non-integral. Yet it seems to me that a victory so won is no victory at all. The advocate of the proof aims to establish that a certain position concerning vagueness is untenable, and on Evans' way of formalising matters the position comes down to holding that $\nabla(a=b)$, $\lambda x[\nabla(x=a)]b$ and $\neg\lambda x[\nabla(x=a)]a$ are simultaneously true. The claim that $a=b$ determinately implies $\lambda x[\nabla(x=a)]a\leftrightarrow\lambda x[\nabla(x=a)]b$ is in itself the claim that this position is *not* tenable. So the attempted proof becomes viciously circular. Shouting ‘Leibniz’s Law!’ does not help.

Noonan (1990, pp.160-62) claims that an opponent of Evans' position is committed to the rejection of a principle he calls the Diversity of the Definitely Dissimilar:

(DDD) $(\forall x)(\forall y)(\Delta Fx\&\Delta Fy\rightarrow((Fx\&\neg Fy)\rightarrow x\neq y))$.

How, asks Noonan, can it reasonably be maintained both that a=b is indeterminate and that there is a property that a definitely has and b definitely does not have? This is a good question. But is someone who holds that lines (1), (2) and (4) of the proof have the value 1 really committed to maintaining such a silly thing? I think not.

To simplify the discussion slightly I will consider in the first instance not (2) and (4) but the forms

(2') $\neg\lambda x[\triangle(x=a)]b$

and (4') $\lambda x[\triangle(x=a)]a$.

(2') asserts that b lacks the property of being determinately identical to a. (4') says that a has the property of being determinately self identical. For the statement that a has the property of being determinately identical to a ascribes one and the same property to a as the statement that a has the property of being determinately identical to itself. It is not as though there are two different properties that a has, the property of being determinately self identical and the property of being determinately identical to a! But b, of course, also has the property of being determinately self identical. So (4') does not attribute to a any property that b lacks. To hold that (2') and (4') are both true is *not* to maintain that a has a property which b lacks.

The situation is the same in the case of (2) and (4). (4) states that a does not have a certain property; and indeed b does not have this property either. To hold that (2) and (4) are both true is not to say that b has a property which a lacks.

The accidental capture of a term by a binding construction is a peril familiar from first order logic. The substitution of x for y in the open sentence $\exists x(x \text{ loves } y)$ is not continuous with other substitutions for y. There is an analogous peril in the case of λ-abstracts containing unbound singular terms. The substitution of a for y and the substitution of b for y in the open sentence $\neg\lambda x[\triangle(x=a)]y$ are not of the same feather. The first substitution produces a statement equivalent in meaning to

$$\neg\lambda x[\triangle(x=x)]a$$

but not the second does not.

This consideration seems to me to hold quite generally and not merely in the case of λ-abstracts containing delta operators. Take, for example, the inference

$$\begin{array}{l} \neg a \in a \\ \underline{b \in a} \\ b \neq a. \end{array}$$

The correct way to represent this reasoning by means of abstracts is

$$\begin{array}{l} \neg\lambda x\lambda y[x \in y]aa \\ \underline{\lambda x\lambda y[x \in y]ba} \\ \qquad b \neq a. \end{array}$$

The attempt to represent the inference by means of monadic abstracts is unsuccessful. There is nothing for $\neg\lambda x[x \in a]a$ to mean other than that $\neg\lambda x[x \in x]a$, there being only one property of a at issue here, namely the property of being a member of itself. Since b also lacks the property of being self-membered, no dissimilarity is established and so one cannot infer $b\neq a$. As Evans' derivation shows, when the delta operators are in play the consequences of ignoring the difference between $\lambda x[Rxa]a$ and $\lambda x\lambda y[Rxy]aa$ can be disastrous.

Evans' derivation can, of course, be recast in relational form. But it is by no means clear that proponents of the thesis whose coherence Evans was trying to challenge must accept each step in the reworked derivation. They may concede that

(2'') $\neg\lambda x\lambda y[\triangle(x=y)]ab$

and (4'') $\lambda x\lambda y[\triangle(x=y)]aa$

together determinately imply $a\neq b$, but decline to agree that (2'') and (4'') follow from anything that they hold to be true. For their position is that there is a single relation, identity, and that it may or may not be indeterminate whether this relation holds; whereas the moves that lead to (2'') and (4'') invite one to eschew the relation of identity simpliciter and countenance in its place two other relations, the relation of being determinately identical, and the relation of being indeterminately identical – a position that the reworked derivation reduces to absurdity.

So, then, is an opponent of Evans' position committed to the rejection of (DDD)? It depends how the notation is intended. If Fx is an invitation to substitute any expression of a certain syntactical form then some instances of (DDD) will have non-integral values, and unproblematically so. If, on the other hand, $Fx\&\neg Fy$ is to be understood as representing the fact that object x has a property that object y lacks then opponents of Evans' position need not reject (DDD) (nor the strengthened form of it in which the final consequent is $x\neq y\&\Delta x\neq y$). As one might put it, the logical form of

$$\lambda x[\nabla(x=a)]b \& \neg\lambda x[\nabla(x=a)]a$$

is not $Fb\&\neg Fa$. Similar remarks apply to Leibniz's Law. To hold that $a=b$ does not determinately imply $\lambda x[\nabla(x=a)]a \leftrightarrow \lambda x[\nabla(x=a)]b$ is not to question the determinate truth of the proposition that if $a=b$ then a and b have all their properties and relationships in common.

III

In his brief prefatory remarks Evans presents his derivation as a challenge to the coherence of the thesis that ‘the world might itself *be* vague’, this thesis being taken to entail that ‘vagueness ... would then be a necessary feature of any true description of [the world]’ (op. cit.). Thus Evans’ target appears to be what is here called the thesis of ontic vagueness. In fact, if Evans’ proof were successful it would show that there can be no vague objects at all, not even of the anodyne sort. I think Sainsbury holds both that there are anodyne vague objects and that Evans’ proof works, but so far as I can see these two views cannot be fitted together.

Sainsbury's favoured way of expressing the thesis of ontic vagueness is (I follow his numbering):

(3) An object is ontically vague iff it satisfies $\lambda z(\nabla\Phi z)x$, for some sharp Φ (p.4).

I see no reason to disagree with Sainsbury when he says that 'identity is sharp, if anything is' (p.7). For if indeterminateness could arise in an identity statement not because the relata are vague objects but because the relation itself is other than sharp then the determinateness of all instances of $\forall x(x=x)$ would be inexplicable.

According to Sainsbury if identity is sharp (3) yields:

If x and y satisfy (9) $\lambda w\lambda z(\nabla(w=z))x,y$ then x and y are ontically vague (p.7).

But by my reckoning

∇(New Devon = Nouvelle Provence)

is true; and to deny the applicability of λ abstraction in this instance would be to maintain, contrary to the position Sainsbury and I both hold, that New Devon, Snowdon and the like are not genuine *objects* with fuzzy boundaries. So Sainsbury's proposed criterion of ontic vagueness deems New Devon an ontically vague object. This seems wrong. There is nothing in the New Devon example to suggest that there is representation-transcendent vagueness.

In my view Sainsbury has given us no reason to think that there is an intelligible way of formulating the dark thought that the world in itself might be vague, and considerable reason to think that there is none.

IV

There is, though, a perfectly intelligible thesis about *representations* that a supporter of ontic vagueness might regard as capturing part at least of what it is they want to say. The fundamental thought of the friend of ontic vagueness seems to be that it isn't (or might not be) *up to us* whether to be vague or not. That is, a believer in ontic vagueness denies what I will call the *crispness postulate*:

> Beyond a certain point of sophistication the physical sciences will use only non-vague, or *crisp*, representations of the world.

To maintain that good science cannot be rid of vague representations is to say that the world constrains us to be vague even where we would most like to be precise. (Notice, incidentally, that the crispness postulate does not exclude quantum mechanical uncertainty principles. It need not always be possible for us to single out one of a crisply specified range of crisp representations as the true one.)

It seems to me that the crispness postulate is an empirical proposition. It is not hard to envisage a possible world in which the postulate is false. Figure 2 is a representation of the Mandelbrot set. The black area represents the set and the grey shaded areas represent the complement of the set.

FIGURE 2 ABOUT HERE

To determine whether a given point c on the Argand plane is in the set, one iterates the expression $x \Leftarrow x^2+c$, starting at $x=0$. If x reaches a certain value known as the *escape radius* - in point of fact $1+\sqrt{2}$ – then the point c is not in the set. If the value of x never reaches the escape radius then the point c is in the set. If a point is *not* in the set this can always be shown in some finite number of iterations, but in general there is no integer n such that if x remains below the escape radius after n iterations then the point *is* in the set. (Things are the other way about in the case of the set of theorems of the predicate calculus. If A *is* a member of the set then this can be shown in some finite number of steps, but the complement of the set is not recursively enumerable.)

The Mandelbrot set is certainly not a vague object: no point on the plane is a borderline case for the predicate 'is a member of the Mandelbrot set'. Figure 2, though, is a vague *representation* of the Mandelbrot set. Many points on the plane are borderline cases for the predicate ‘is represented by the figure as being in the

Mandelbrot set'. Indeed, the figure is *visibly* fuzzy. It has always seemed remarkable to me that modern discussions of vagueness should dwell exclusively on sentential representations and ignore pictorial and other geometrical representations. Things were not always this way. In his seminal paper 'Vagueness' Bertrand Russell wrote

> Vagueness ... is a conception applicable to every kind of representation - for example, a photograph, or a barograph. ... [A] photograph which is so smudged that it might equally represent Brown or Jones or Robinson is vague. ... I think all vagueness in language and thought is essentially analogous to this vagueness which may exist in a photograph. (1923, pp.89, 91)

The boundary between the Mandelbrot set and the complement of the set is an example of a so-called fractal boundary (Mandelbrot 1977). The point I want to emphasise is that no fragment of the boundary has a crisp finite representation. Zoom in on any tiny segment of the boundary and a wealth of additional detail is revealed (figure 3). Zoom in again and the same thing happens (figure 4). (The black areas represent parts of the set and the grey areas and foamy swirls represent parts of the complement.) The process could be repeated *ad infinitum*: the fantastic detail of the boundary is bottomless.

FIGURES 3 AND 4 ABOUT HERE

There is no such thing as a crisp finite representation of the Mandelbrot set. Switching to sentential representations does not help. No finite sentence nor finite set of finite sentences nor even a recursive set of finite sentences can be a crisp and accurate representation of the Mandelbrot set, for there will be truths about the set that are not entailed by the sentences in question. Finite beings must make do with vague representations of the set. (The instructions I gave for *generating* the set are certainly crisp and were expressed in a finite number of words but, as Turing showed us, a finite instruction for generating an object is not the same thing as a finite representation of the object. The instructions I gave call for an infinite number of steps to be performed in the course of generating the complete membership of the set.)

The crispness postulate is false in any universe that is – as Russell's antagonist in the well-known anecdote might put it – Mandelbrots all the way down.[6] There is no reason to think that the real world contains any boundaries like the one between the Mandelbrot set and its complement, but this is not the only way in which Mother Nature might resist crisp characterisation by finite means. Certainly it is an empirical question whether our universe is so characterisable.[7]

Notes

1. Although for a contrary view see Tye 1990, p.556.

2. See Pelletier 1989, p.482, note 4.

3. From time to time three-valued ‘vague logics’ have appeared (for example Broome 1984 and Johnsen 1989). The three-valued approach has little to recommend it. All indeterminate statements are thrown indiscriminately into the same box. This austere treatment wastes much significant information, for example that A is truer than B, or that C is nearly but not completely true, or that drawing a given inference will produce a conclusion that is no less true than the weakest premiss. A three-valued approach cannot generate a useable extension of the classical theory of inference. In practical terms the effect of assigning a statement the catch-all value Neither is simply to bar it from functioning as a premiss in inference.

4. These clauses for ¬, & and → are essentially those used by Lukasiewicz in his three valued logic (Lukasiewicz 1920). It seems that Prior was the first to consider infinite-valued extensions of Lukasiewicz's three-valued matrices (Prior 1957, pp.22-24).

5. Evans does not display his derivation of (5') from the strengthened form of Leibniz's Law , saying only that it involves prefixing each of lines (1)-(4) by Δ. Presumably the derivation proceeds by means of applying the K-like ‘axiom’

$$(K\Delta) \qquad \Delta(A\rightarrow B)\rightarrow(\Delta A\rightarrow \Delta B)$$

to the strengthened form of Leibniz’s Law. However, (KΔ) is not valid in the present semantics: where $v(A)=0$ and $v(B)$ is non-integral, (KΔ) takes the value 0. (See also Pelletier 1984, p.417.)

6. See Hawking 1988, p.1.

7. With thanks to Diane Proudfoot for much extremely helpful discussion, to Philip Catton, Dominic Hyde, Mark Sainsbury, Peter Smith, Roy Sorensen, Michael Tye and Tim Williamson for comments and suggestions, and to Craig Webster for figures 2, 3 and 4.

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