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### **Vague Identity and Fuzzy Logic\***

#### ABSTRACT

Fuzzy logic extends deductive methods to situations in which the information available may be only partly or approximately true. Fuzzy logic has often been championed as a logic of vague terms, and it does indeed provide an intuitive analysis of what goes wrong in Sorites reasoning. Here a fuzzy semantics is given for a language containing the quasi-modal operators 'Determinately' ( $\Delta$ ) and 'Indeterminately' ( $\nabla$ ). The semantics is sensitive to higher-order vagueness. For example, the semantics distinguishes between Herbert's being a clear borderline case of a bald man and his being a borderline borderline case of a bald man. I show that a famous reductio ad absurdum of the statement ' $\nabla(a=b)$ ', due to Gareth Evans, is not valid when the background logic is fuzzy logic. Moreover, an improved form of Evans's reductio due to Harold Noonan is also not valid.

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In a five-line derivation Gareth Evans reduced to absurdity the assumption that some identity - any identity - is indeterminate, or so he claimed.<sup>1</sup> His notation "allow[s] for the expression of the idea of indeterminacy by the sentential operator ' $\nabla$ '" (*ibid.*, p. 208). The derivation runs as follows.

- (1)  $\nabla(a=b)$
- (2)  $\lambda x[\nabla(x=a)]b$  from (1)
- (3)  $\sim \nabla(a=a)$  axiom
- (4)  $\sim \lambda x[\nabla(x=a)]a$  from (3)
- (5)  $a \neq b$  from (2) and (4) by Leibniz's Law.

Evans remarked that (5) "contradict[s] the assumption, with which we began, that the identity statement ' $a=b$ ' is of indeterminate truth value" (*ibid.*).

The reason for employing Church's lambda notation for property abstraction (whereby the statement 'a has the property F' is represented ' $\lambda x[Fx]a$ ') is brought out by the following simplified version of the derivation:

- (1a)  $\nabla(a=b)$
- (2a)  $\sim \nabla(a=a)$  axiom
- (3a)  $a=b \rightarrow (\sim \nabla(a=a) \rightarrow \sim \nabla(a=b))$  from Leibniz's Law
- (4a)  $a \neq b$  from (1a), (2a) and (3a) by prop. calc.

Line (3a) is an obvious weak point, for it is well known that in contexts containing (what Prior called) quasi-modal operators, of which  $\nabla$  is an example, the substitution of identicals can turn a truth to a falsehood. Much more compelling is the analagous form of (3a) in which property abstraction is employed (assuming, of course, that one is prepared to countenance *being indeterminately identical to a* as a genuine property):

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<sup>1</sup>Evans, G. 'Can There Be Vague Objects?' *Analysis*, 38 (1978): 208.

(3b)  $a=b \rightarrow (\sim \lambda x[\forall(x=a)]a \rightarrow \sim \lambda x[\forall(x=a)]b)$ .

In association with lines (2) and (4) of the original derivation, (3b) yields the desired conclusion.

Notoriously, Evans's brief paper left unclear the precise moral he wished to draw from this supposed reductio. Although he did not say so in the paper his target was in fact Michael Dummett's view that the world *in itself* might be vague. Evans makes this clear in a letter written to David Lewis in November 1978:

The author I had in mind was Dummett, who has written and spoken of vagueness being in the world, and so on, but I did not refer to it because then as now I cannot find a reference in his published writings.<sup>2</sup>

Perhaps this passage would have served Evans's purpose:

I [said] in 'Wang's Paradox'<sup>3</sup> [that] 'the notion that things might actually *be* vague, as well as being vaguely described, is not properly intelligible' [p. 314]. I should now like to withdraw this remark. ... [I]t is natural to us to conceive of physical reality as, in itself, capable of description in absolutely precise mathematical terms ... This inclination does not, however, appear to be more than a prejudice ... It is not apparently absurd to suppose the contrary, namely that the physical world is in itself such that the most precise description of it that even omniscience would yield might

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<sup>2</sup>I am grateful to Antonia Phillips and to the Gareth Evans Memorial Trust for permission to quote from this letter and to David Lewis for information and assistance.

<sup>3</sup>Dummett, M.A.E. 'Wang's Paradox' *Synthese*, 30 (1975): 301-324.

yet involve the use of expressions having some degree of vagueness.<sup>4</sup>

In the opening sentences of his paper Evans gives a tolerably clear statement of the Dummettian position that he wishes to challenge:

It is sometimes said that the world might itself *be* vague. Rather than vagueness being a deficiency in our mode of describing the world, it would then be a necessary feature of any true description of it. (*op. cit.*, p. 208)

Regrettably, Evans gives his readers no detailed guidance as to how his derivation is intended to bear on this position.<sup>5</sup> However, leaving aside the dark issue of vagueness-in-the-world, what can be said is that Evans's derivation presents a *prima facie* challenge to the view that ' $\nabla(a=b)$ ' is true. Indeed, his paper is not uncommonly regarded as presenting a major

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<sup>4</sup>Dummett, M.A.E. *The Interpretation of Frege's Philosophy* London: Duckworth (1981), p. 440.

<sup>5</sup>Exegesis is offered by Burgess, J.A. 'Vague Identity: Evans Misrepresented' *Analysis*, 49 (1989): 112-119; Garrett, B. 'Vagueness and Identity' *Analysis*, 48 (1988):130-134, 'Vague Identity and Vague Objects' *Noûs*, 25 (1991): 341-351, and 'Vagueness, Identity, and the World' *Logique et Analyse*, 135-136 (1991): 349-358; Lewis, D.K. 'Vague Identity: Evans Misunderstood' *Analysis*, 48 (1988): 128-130; Noonan, H.W. 'Indefinite Identity: A Reply to Broome' *Analysis*, 44 (1984): 117-21 and 'Vague Identity Yet Again' *Analysis*, 50 (1990): 157-62; Over, D.E. 'Vague Objects and Identity' *Analysis*, 49 (1989): 97-99; Pelletier, F.J. 'Another Argument Against Vague Objects' *Journal of Philosophy*, 86 (1989): 481-92.

objection - perhaps the principal objection - to the view that there are vague or fuzzy or indeterminate identities.<sup>6</sup>

Evans's derivation falls short of being a reductio, for there is no explicit contradiction between (1) and (5). Evans seemed to think that the derivation can be extended to yield an explicit contradiction. He wrote:

If 'Indefinitely' and its dual, 'Definitely' (' $\Delta$ '), generate a modal logic as strong as S5, (1) - (4) and, presumably, Leibniz's Law, may each be strengthened with a 'Definitely' prefix, enabling us to derive:

(5')  $\Delta \sim(a=b)$ . (*op. cit.*, p. 208)

It has often been remarked that the reasoning here is hard to reconstruct, for if ' $\nabla$ ' and ' $\Delta$ ' are duals then ' $\Delta(a=b)$ ' says that ' $a \neq b$ ' is not indeterminate, and so is true if ' $a=b$ ' is determinately false. (A pair of quasi-modal operators  $\uparrow$  and  $\downarrow$  are *duals* just in case each is definable in terms of the other in the fashion of  $\square$  and  $\diamond$ , viz  $\uparrow A \equiv \sim \downarrow \sim A$  and  $\downarrow A \equiv \sim \uparrow \sim A$ .) Yet if the logic of ' $\Delta$ ' is 'as strong as S5' then ' $\Delta(a=b)$ ' implies ' $a=b$ '.<sup>7</sup> (In the letter previously mentioned, Evans himself agrees that the falsity of 'P' suffices for the truth of ' $\Delta P$ '.)

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<sup>6</sup>For example by Broome, J. 'Indefiniteness in Identity' *Analysis*, 44 (1984): 6-12, p. 7; Burgess, J.A. 'Vague Objects and Indefinite Identity' *Philosophical Studies*, 59 (1990): 263-287, p. 268; Gibbins, P.F. 'The Strange Modal Logic of Indeterminacy' *Logique et Analyse*, 100 (1982): 443-446, p. 446; Johnsen, B. 'Is Vague Identity Incoherent?' *Analysis*, 49 (1989): 103-112, p. 106; Lowe, E.J. 'Vague Identity and Quantum Indeterminacy' *Analysis*, 54 (1994): 110-114, p. 111; Parsons, T. 'Entities Without Identity' *Philosophical Perspectives*, 1 (1987): 1-19, p. 12; Tye, M. 'Vague Objects' *Mind*, 99 (1990): 535-57, p. 555.

<sup>7</sup>See also notes 31 and 44.

However, this slip of Evans's does not matter much, for even in the absence of an explicit contradiction, the derivation (1) - (5) is trouble enough for the view that there may be indeterminate identities. I will use ' $\vdash$ ' to represent the relation of deductive consequence associated with the proof. If the proof is not fallacious, we have

$$\nabla(a=b) \vdash a \neq b.$$

Contraposing and eliminating the double negation gives

$$a=b \vdash \sim \nabla(a=b).^8$$

The bearing of a result such as this concerning identity *statements* on the Dummettian claim that there might be vagueness in the world itself is far from clear. Nevertheless, this result, if correct, is a substantial one in its own right. Assuming that we are prepared to apply detachment to what we assert, the result tells us that vague identity statements are assertible only on pain of inconsistency: if it is the case that  $\nabla(a=b)$  then asserting ' $a=b$ ' leads, via detachment, to the contradiction  $\nabla(a=b) \& \sim \nabla(a=b)$ . (Or if the principle derived in note **eight** is employed then asserting ' $a=b$ ' when the latter is not determinate leads to the contradiction

$\Delta(a=b) \& \sim \Delta(a=b)$ .) This is just the sort of result that an opponent of vague identity needs, for it drives a wedge between identity and concepts that are uncontroversially vague, such as *bald* and *tall* and *blue*. As an admirer of the

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<sup>8</sup>The validity of contraposition in contexts involving ' $\nabla$ ' has been questioned (for example by Parsons *op. cit.*, pp. 9-10). Indeed, Evans's whole derivation has a contrapositive flavour. A more direct version, free of contrapositions, leads to the pleasingly blunt  $a=b \vdash \Delta(a=b)$ . The derivation is: (1)  $a=b$  (assumption); (2)  $\Delta(a=a)$  (axiom); (3)  $\lambda x[\Delta(x=a)]a$  (from (2)); (4)  $\lambda x[\Delta(x=a)]b$  (from (1) and (3) by Leibniz's Law); (5)  $\Delta(a=b)$  (from (4) by  $\lambda$ -elimination).

vagueness of ordinary language might put it, the practical value of a vague language is that speakers can assert what is *true enough* for their purposes. If Evans's result is correct, it shows that in the case of identity, nothing less than determinate truth can be truth enough. But is the result correct?

Fuzzy logic is widely canvassed as the logic of vagueness. Yet no-one, I think, has explored the question of whether or not Evans's derivation can be carried through in fuzzy logic. I shall argue that it cannot.<sup>9</sup> (Several articles have investigated the fate of Evans's derivation in a three-valued setting, but three-valued logic is not fuzzy logic.<sup>10</sup>)

Fuzzy logic aims to extend ordinary deductive methods to situations in which the information available may be only partly or approximately true. In fuzzy logic each statement has some numerical *degree* of truth or falsity. These degrees are the truth-values of the logic, and for full generality continuum many values are usually permitted. The three-valued approach, with its one grudging non-classical value, has little to recommend it in the context of vagueness. With only three values to hand, all statements that are neither completely true nor completely false have to be thrown indiscriminately into the Neither box. This wastes much significant information, for example that A is truer than B, or that C is nearly but not completely true, or that drawing a given inference will produce a conclusion that is no less true than the weakest premiss. The three-valued approach cannot generate a useable extension of the classical theory of inference, for in practical terms the effect of assigning a statement the catch-all value Neither

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<sup>9</sup>See also my 'On Vague Objects, Fuzzy Logic and Fractal Boundaries' *Southern Journal of Philosophy*, supp. vol. 33 (1994): 83-96.

<sup>10</sup>For example Broome *op. cit.*, Johnsen *op. cit.*, Parsons *op. cit.*

is simply to prevent us from usefully employing it as a premiss in inference. It is of no practical use whatever to be told (as Lukasiewicz tells us) that if either one or the other premiss of an application of modus ponens has the value Neither then the conclusion is either True or Neither.<sup>11</sup> Whereas the information that if both premisses of an application of modus ponens are approximately true then so is the conclusion is exactly the sort of thing one needs to know in order to be able to reason with vague statements.

To fill in some formal detail, let every statement under consideration have a value lying in the closed interval  $[0, 1]$  of the real numbers. ("Closed" means simply that the end-points 0 and 1 are included in the interval.) I will call 0 and 1 the *integral* values. 0 represents complete or determinate falsehood, 1 complete or determinate truth. Identity statements are, of course, among those that may take non-integral values. Truth conditions are given for compound statements in terms of arithmetical operations on values. For example:

$$v(\sim A) = 1 - v(A)$$

('-' represents ordinary subtraction)

$$v(A\&B) = \min (v(A), v(B))$$

( $\min (i, j)$  is the smaller of the two numbers  $i$  and  $j$ )

$$v(A\vee B) = \max (v(A), v(B))$$

( $\max (i, j)$  is the larger of  $i$  and  $j$ )

$$v(A\rightarrow B) = 1 \text{ if } v(B) \geq v(A); v(A\rightarrow B) = 1 - (v(A) - v(B)) \text{ if } v(B) < v(A)$$

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<sup>11</sup>Lukasiewicz, J. 'On Three-Valued Logic' (1920), p. 88, and 'Philosophical Remarks on Many-Valued Systems' (1930), p. 166; both in Borkowski, L. (ed.) *Jan Lukasiewicz: Selected Works* Amsterdam: North Holland (1970).

$$v(A \leftrightarrow B) = \min (v(A \rightarrow B), v(B \rightarrow A)).^{12}$$

(Lukasiewicz himself took the non-integral values of his semantics to be degrees of probability rather than degrees of truth (1922: 130).)

It is natural to think of  $\Delta A$  as being true (determinately) just in case  $A$  has an integral value and as being false (determinately) otherwise:

$$v(\Delta A) = 1 \text{ iff either } v(A) = 1 \text{ or } v(A) = 0$$

$$v(\Delta A) = 0 \text{ iff } 0 < v(A) < 1.$$

It is equally natural to think of  $\nabla A$  as being true (determinately) just in case  $A$  has a non-integral value and as being false (determinately) otherwise:

$$v(\nabla A) = 1 \text{ iff } 0 < v(A) < 1$$

$$v(\nabla A) = 0 \text{ iff either } v(A)=1 \text{ or } v(A)=0.$$

Notice that the delta operators are duals, as Evans requires.

Fuzzy logic's detractors scoff at the "excess precision" afforded by the presence of continuum-many degrees of truth (for example Tye).<sup>13</sup> What, they ask, has this bizarre sensitivity got to do with actual reasoning in the real world? This is largely to miss the point. It is only in the maximally general mathematical theory that infinitely many values are considered. Compare this "objection" to the general mathematical theory of computation: What does a theory that countenances infinitely many machines with tapes of unbounded length have to do with actual computation in the real world? In practical applications - for example a logic of vagueness for implementation in

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<sup>12</sup>Lukasiewicz, J. 'A Numerical Interpretation of the Theory of Propositions' (1922) (in Borkowski, *op. cit.*); Zadeh, L.A. 'Fuzzy Logic and Approximate Reasoning' *Synthese*, 30 (1975): 407-28.

<sup>13</sup>Tye, M. 'Vagueness: Welcome to the Quicksand' *Southern Journal of Philosophy*, supp. vol. 33 (1994): 1-22, p. 11.

an AI reasoning program, or a fuzzy expert system for controlling some industrial process - only a finite number of non-integral values will be employed. The general theory is an *idealisation*, abstracted from actual engineering practice (or, as it was in the first instance, from intended practice). Real engineers typically use the non-integral values to *rank* propositions according to their relative degree of truth, rather than to ascribe absolute degrees of truth. As Goguen says, "we should not expect particular numerical values ... to be meaningful (except 0 and 1), but rather their *ordering*".<sup>14</sup> The spectrum of non-integral values is thought of as roughly (and fuzzily) partitioned into regions that bear linguistic labels such as 'very true', 'nearly true', 'more or less true', 'barely more true than false', 'more false than true', 'almost completely false', and so forth. Thus to an engineer the assignments  $v(p) = 0.9$ ,  $v(q) = 0.95$  may indicate simply that p and q are both pretty true, and that q is a bit truer than p. Since the ranking is open, in the sense that more propositions may come along later and occupy intermediate positions, the engineer must have access to values *between* any two values, at any rate up to some resource limit. Generally fuzzy logicians take the real line to be the appropriate idealisation, although my own preference is for the rationals rather than the reals.

One reason for the suggestion that fuzzy logic is the logic of *vagueness*, rather than merely a logic of *inexactness*, is that in fuzzy logic the sorites paradoxes melt away.<sup>15</sup> The major premiss of any sorites, while no doubt

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<sup>14</sup>Goguen, J.A. 'The Logic of Inexact Concepts' *Synthese*, 19 (1969): 325-373, p. 332.

<sup>15</sup>An example of a statement that is inexact but not vague is one recording a numerical measurement that is accurate only to two significant figures, the

true enough for any conceivable practical purpose, is not completely true. Using the major premiss once, or even several times, produces no difficulty. However, at each step the statement that is inferred deviates from complete truth, and this deviation, while tiny at first, increases relentlessly with each application of modus ponens, until after enough iterations the conclusion is gapingly false.

This is illustrated by the classical paradox of the bald man (said to originate with Eubulides).

(1)  $\forall n(x \text{ is not bald when he has } n \text{ hairs on his head} \rightarrow x \text{ is not bald when he has } n-1 \text{ hairs on his head})$

(2)  $x \text{ is not bald when he has a million hairs on his head}$

∴ (3)  $x \text{ is not bald when he has } 0 \text{ hairs on his head.}$

(1) is not completely true (for the sorites argument reduces to absurdity the supposition that (1) is completely true).<sup>16</sup> Let its value be  $1 - \delta$ , for some  $\delta > 0$ . The idea is that for practical purposes,  $\delta$  is negligible. Let us assume, for the moment, that the value of each elimination instance of a universal quantification,  $\forall xP$ , is identical to the value of  $\forall xP$  itself. I will represent 'x

quantity measured being continuous. For a representation to be vague there must be borderline cases of its applicability.

<sup>16</sup>As is well known, the remaining position, that (1) is completely false, has a high price. For then

$\exists n(x \text{ is not bald when he has } n \text{ hairs on his head} \ \& \ x \text{ is bald when he has } n-1 \text{ hairs on his head})$

is completely true. The supposition that there is such an  $n$  strikes many as fantastic.

is bald when he has  $n$  hairs on his head' by ' $B_n$ '. The value of the conclusion of the first step of the sorites is obtained by solving the equation

$$1 - \delta = 1 - (1 - v(\sim B_{999,999}))$$

and of the second step,  $\sim B_{999,998}$ , by solving

$$1 - \delta = 1 - ((1 - \delta) - v(\sim B_{999,998}))$$

and of the third step by solving

$$1 - \delta = 1 - ((1 - 2\delta) - v(\sim B_{999,997}))$$

and so on. The values of the members of the sorites sequence  $\sim B_{1,000,000}$ ,  $\sim B_{999,999}$ , . . . descend steadily toward 0, each decrement being by an amount that is locally negligible:

$\sim B_{1,000,000}$	1
$\sim B_{999,999}$	$1 - \delta$
$\sim B_{999,998}$	$1 - 2\delta$
$\sim B_{999,997}$	$1 - 3\delta$
.	.
.	.
.	.

Each iteration of modus ponens is truth-preserving *enough*, in that the value of the conclusion always differs by only a negligible amount from the value of the conjunction at the premisses. But chain enough steps together, and a quantity that is negligible after a single iteration will start to make its presence felt. This strikes me as a revealing theoretical reconstruction of the sorites phenomenon. It fits well with what one is inclined to say pretheoretically about a sorites: each individual step looks all right yet somehow disaster results.

At any rate, those are the basic ideas. The story is not yet completely correct, for it is surely counterintuitive that  $\sim B_{999,999}$  should be ranked as

*less true* than  $\sim B_{1,000,000}$ . These two statements are both completely true: neither is truer than the other, not even by a tiny amount. That the foregoing story holds otherwise is a consequence of the simplifying assumption that the value of *every* elimination instance of  $\forall n(\sim B_n \rightarrow \sim B_{n-1})$  is  $1 - \delta$ . More plausibly, restricting the quantifier in different ways will produce formulae of differing values. In particular, there will be formulae of the form  $\forall n > j(\sim B_n \rightarrow \sim B_{n-1})$  that are completely true. Of course, it is in the nature of things that we cannot specify the least  $j$  for which this is so, and nor can we specify the greatest  $k$  such that  $v(\forall n < k(\sim B_n \rightarrow \sim B_{n-1})) < 1$ . The graph of  $v(\sim B_n)$  against  $n$  consists of a number of segments of different gradients. Up in the high figures the line is flat (certainly the value of  $\sim B_{1,000,000}$ ,  $\sim B_{999,999}$ ,  $\sim B_{999,998}$  and  $\sim B_{999,997}$  is in each case 1). No segment quite meets its neighbour: the point  $(0, \sim B_0)$  excepted, lines begin and end in a blur. Within these fuzzy regions there are no crisp facts of the matter concerning the values or the derivatives of the function  $f(n) = v(\sim B_n)$  (unless one is an epistemic theorist, in which case the segments of the graph do meet, but always beneath a blot of ignorance). These fuzzy regions on the graph are regions of *higher-order* indeterminacy, a matter to which I will return shortly.

The idea of using fuzzy logic to resolve sorites paradoxes has appealed to a number of recent writers. There seems to be a broad consensus among these writers that - contrary to what I am suggesting here - the precise culprit is modus ponens: the inference

$$\sim B_n \rightarrow \sim B_{n-1}, \sim B_n \vdash \sim B_{n-1}$$

is invalid, it is said.<sup>17</sup> The thought is that an inference is valid if and only if the value of the conclusion can be *no less than* the value of the weakest premiss; and, as we have seen, there are many points in the sorites sequence at which the value of the premiss  $\sim B_n$  exceeds the value of  $\sim B_{n-1}$ . One advocate of fuzzy logic who has objected to this treatment of modus ponens is Edgington. She sensibly remarks: "To say that [modus ponens] is invalid obscures an important distinction: between the case where the fall in value of the conclusion is constrained by the values of the premisses, and the case where it is not".<sup>18</sup>

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<sup>17</sup>See, for example, Forbes, G. *The Metaphysics of Modality* Oxford: Clarendon Press (1985), pp. 171-2 and *Modern Logic* New York: Oxford University Press (1994), pp. 355-6; Peacocke, C. 'Are Vague Predicates Incoherent?' *Synthese*, 46 (1981): 121-141, p. 127; Sainsbury, R.M. *Paradoxes* Cambridge: Cambridge University Press (1988), pp. 42-3. (Sainsbury there maintains that the fuzzy logician "must challenge the validity of modus ponens" (p. 42). In the second edition of the book (1995), he says (persuaded by Edgington, D. 'Validity, Uncertainty and Vagueness' *Analysis*, 52 (1992): 193-203): "degree theory need give us no reason to say that modus ponens is not valid" (p. 42). However, he still thinks that any fuzzy logician who *does* deny the validity of Sorites reasoning (as opposed to denying the soundness of the reasoning on the ground that the premisses are not fully acceptable) must deny the validity of modus ponens (pp. 41-42).) On the analysis I present below, it is the invalidity of the classical rule of contraction, and not of modus ponens, that accounts for the invalidity of the Sorites.

<sup>18</sup>Edgington *op. cit.*, p. 200; see also her 'Wright and Sainsbury on Higher-order Vagueness'. *Analysis*, 53 (1993): 193-200.

The all-or-nothing distinction between the logically perfect and the logically worthless that motivates the classical account of argument validity is completely foreign to fuzzy logic. Here the engineer's principle of "right is right enough" holds sway. Each application of modus ponens in the sorites argument is good enough, since the extent to which the value of the conclusion deviates below the value of the weakest premiss never exceeds  $\delta$ , and by hypothesis  $\delta$  is negligible. In general, where  $A \rightarrow B$  is nearly true, B's value, although less than A's, will always be nearly the same as A's. Modus ponens is truth preserving enough.

In fuzzy logic, the unrestricted iteration of valid inferences may lead to trouble. A detailed syntactical analysis of the reasoning involved in the paradox of the bald man will both pinpoint the problem and afford a way of dealing with it. (The semantics of the situation must await another occasion.) I will use the proof-theoretic apparatus introduced by Gentzen, and in particular his cut rule, or transitivity rule.<sup>19</sup> A Roman letter represents a single formula and a Greek letter represents either a single formula, or a string of formulae separated by commas, or the null string.

$$\text{Cut} \quad \frac{\Gamma \vdash \Theta, A \quad A, \Pi \vdash \emptyset}{\Gamma, \Pi \vdash \Theta, \emptyset}$$

Premiss (1) of the sorites argument, the major premiss, will be abbreviated by the letter 'M'. The first iteration in the chain of reasoning can be represented like this:

$$M \vdash \sim B_{1,000,000} \rightarrow \sim B_{999,999}$$

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<sup>19</sup>Gentzen, G. 'Investigations Into Logical Deduction' (1934); in Szabo, M.E. (ed.) *The Collected Papers of Gerhard Gentzen* Amsterdam: North-Holland (1969).

$$\sim B_{1,000,000} \rightarrow \sim B_{999,999}, \sim B_{1,000,000} \vdash \sim B_{999,999}$$

$$M, \sim B_{1,000,000} \vdash \sim B_{999,999} \quad \text{by cut}$$

The second iteration is:

$$M \vdash \sim B_{999,999} \rightarrow \sim B_{999,998}$$

$$\sim B_{999,999} \rightarrow \sim B_{999,998}, \sim B_{999,999} \vdash \sim B_{999,998}$$

$$M, \sim B_{999,999} \vdash \sim B_{999,998} \quad \text{by cut}$$

Amalgamating the results of these two iterations by a further application of cut produces:

$$M, M, \sim B_{1,000,000} \vdash \sim B_{999,998}.^{20}$$

Notice that the cut rule is very picky. It insists that  $\Gamma$  and  $\Pi$ , whatever they may be, are written out in full on the left hand side of the line that is being derived. The rule makes no provision for doing away with the second occurrence of  $M$  so generated. In classical logic, any repetitions of a formula, once obtained, may be deleted by an application of the separate rule of *contraction*:

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}$$

$$A, \Gamma \vdash \Delta$$

However, this cannot be done here. The fuzzy logic that I am advocating is a *contraction-free* logic.

At the end of the chain of inferences that make up the sorites, we arrive at:

$$M, M, \dots, M, \sim B_{1,000,000} \vdash \sim B_0.$$

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<sup>20</sup>Aficionados of the sequent calculus will see that exchanges are being suppressed (for simplicity of presentation).

(To be precise, there are 1,000,000 occurrences of the letter 'M'.) In the absence of unrestricted contraction, this cannot be transformed into the unwanted

$$M, \sim B_{1,000,000} \vdash \sim B_0.$$

Think of it this way. The number of occurrences of 'M' is a record of the number of iterations of modus ponens. Where the number of iterations is small, contraction may safely be applied (always assuming the situation is such that each iteration is roughly truth-preserving, in the sense explained previously: the value of the derived formula never deviates by more than some small fixed amount from the value of the conjunction of the two formulae from which it is derived). But where a statement of derivability is flagged with a long sequence of repetitions, contraction is dangerous.

The flagging formula, in this case M, need not be a universal quantification. For example, a conjunction of conditionals will serve just as well. The point is that applications of cut can be used to chain together a series of instances of modus ponens only if some formula is found with which progressively to flag each successive result of applying cut.<sup>21</sup> One notationally economical way to achieve the flagging in the case where the sorites reasoning has a series of conditional premisses in place of M is to use the antecedent of the first application of modus ponens as the flagging formula. (The conditional itself would serve instead, but is longer.) The idea is that P is used once in the inference

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<sup>21</sup>Gentzen's rule of *weakening* may be used to introduce occurrences of the flagging formula:

$$\frac{A, \Gamma \vdash \Delta}{A, A, \Gamma \vdash \Delta}$$

$P, P \rightarrow Q \vdash Q,$

and then *again* in using  $Q$  in the inference

$Q, Q \rightarrow R \vdash R,$

and so on. Thus combining these two inferences in a way that keeps track of the number of times  $P$  is used will yield

$P, P, P \rightarrow Q, Q \rightarrow R \vdash R.$

(Imagine that whenever a formula is employed as a premiss of a step in the reasoning, it itself, together with each formula higher in the derivation upon which it depends, lights up. The notation keeps track of the number of times that  $P$  lights up.) In a contraction-free logic, these repetitions of  $P$ , once introduced, can be eliminated only in special circumstances.<sup>22</sup>

Contraction-free logics were first studied by Arthur Prior and Carew Meredith in the 1950s. A punting trip in Oxford in the summer of 1956 was the start of a collaboration between Prior, Meredith, his cousin David Meredith, John Lemmon, and Ivo Thomas, which ensued in the legendary joint paper "Calculi of Pure Strict Implication". This circulated in dittoed form

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<sup>22</sup>A crude first approximation to the semantics of the situation is this. The value of the antecedent of any sequent is the minimum of the values of the individual antecedent formulae *except that* before the minimum is calculated the value of any repeated formula is multiplied by itself  $n$  times, where  $n$  is the number of occurrences among the antecedent formulae of the formula in question. For example, the value of the antecedent of the sequent  $p, p, q \vdash r$  is  $\min (v(p) \times v(p), v(q))$ .

for many years.<sup>23</sup> In 1963 a second paper appeared, "Notes on the Axiomatics of the Propositional Calculus", written by Prior and Carew Meredith.<sup>24</sup> Section 7 of the paper gives details of a contraction-free combinatory logic first proposed by Meredith in 1956 or earlier.<sup>25</sup> In their romp through weak implicational logic, Prior and Meredith doubtless gave little thought to practical applications of the systems they discovered.<sup>26</sup> Nowadays contraction-free logic is the object of considerable attention in computer science, largely because of the remarkable fact that if contraction is forbidden, then first-order predicate logic is decidable.<sup>27</sup> Among the

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<sup>23</sup>It was eventually published as Lemmon, E.J., Meredith, C.A., Meredith, D., Prior, A.N., Thomas, I. 'Calculi of Pure Strict Implication' in Davis, J.W., Hockney, D.J., Wilson, W.K. (eds) *Philosophical Logic*. Dordrecht: Reidel (1969).

<sup>24</sup>Meredith, C.A., Prior, A.N. 'Notes on the Axiomatics of the Propositional Calculus' *Notre Dame Journal of Formal Logic*, 4 (1963): 171-187.

<sup>25</sup>An unpublished typescript of Prior's gives the date of Meredith's system as ' $\leq 1956$ '.

<sup>26</sup>It is interesting that, in addition to the various applications of contraction-free logic that have subsequently arisen, the condensed detachment rule that Meredith devised (Meredith and Prior, *op. cit.*) is the basis of the Resolution Principle employed in most current approaches to automated theorem proving (Bunder, M. 'Logics Without Contraction II'; see also my 'Prior's Life and Legacy'; both are in my *Logic and Reality: Essays on the Legacy of Arthur Prior* Oxford: Oxford University Press (1996)).

<sup>27</sup>See, for example, Ketonen, J., Weyhrauch, R. 'A Decidable Fragment of Predicate Calculus' *Theoretical Computer Science*, 32 (1984): 297-307. The reason is that Gentzen's subformula property yields a decision method for

computer science community, probably the best-known contraction-free logic is Girard's linear logic.<sup>28</sup> Girard aptly describes the contraction rule as permitting the use of resources *ad libitum*: in the presence of contraction, a premiss P can be used any number of times in the course of obtaining a derivability-statement that shows only *one* occurrence of P on the left.<sup>29</sup> Various other contraction-free logics are under study, particularly in Australasia, notably by Ross Brady, by Prior's former student Robert Bull, and by Martin Bunder.<sup>30</sup> Fuzzy logic is a pleasant addition to the family.

Turning once more to the issue of vague identity and Evans's derivation, the first point to notice is that in fuzzy logic, inference by Leibniz's Law is not always correct. Where  $a=b$  is indeterminate, an inference

$$\emptyset(a) \vdash \emptyset(b)$$

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predicate logic provided there is a known bound to the length of the sequents involved in a cut-free proof of any given formula, and in the absence of contraction this bound is simply the length of the formula to be proved. (The subformula property is as follows: the only formulae that occur in a cut-free proof of any given sequent  $\Gamma \vdash \Pi$  are subformulae of the formulae in  $\Gamma$  and  $\Pi$  (Gentzen *op. cit.*, pp. 87-88).) Incidentally, there is no presumption that the cut rule can be *eliminated* from the proof theory of the fuzzy logic here considered.

<sup>28</sup>Girard, J.Y. 'Linear Logic' *Theoretical Computer Science*, 50 (1987): 1-102.

<sup>29</sup>Girard, J.Y. 'Geometry of Interaction' (1988), unpublished, p. 13.

<sup>30</sup>Brady, R. 'Gentzenization and Decidability of Some Contraction-less Relevant Logics' *Journal of Philosophical Logic*, 20 (1991): 97-117; Bull, R. 'Logics Without Contraction I', in my *Logic and Reality: Essays on the Legacy of Arthur Prior* Oxford: Oxford University Press (1996); Bunder, M. *op. cit.*

may fail, for the value of  $\emptyset(a)$  may be 1 and yet the value of  $\emptyset(b)$  be non-negligibly less than 1. Evans's own references to Leibniz's Law seem to have been to the conditionalised form

$$a=b \rightarrow (\emptyset(a) \leftrightarrow \emptyset(b)),$$

for he speaks, in a passage quoted earlier, of strengthening Leibniz's Law with a 'Definitely' prefix.<sup>31</sup> But there is nothing law-like about this conditional in the present semantics. When the value of 'a=b' drops below 1 the value of

$$\lambda x[\nabla(x=a)]a \leftrightarrow \lambda x[\nabla(x=a)]b$$

becomes 0 (since the value of the left-hand formula is 0 and the right hand formula 1), and so the value of

$$a=b \rightarrow (\lambda x[\nabla(x=a)]a \leftrightarrow \lambda x[\nabla(x=a)]b)$$

becomes non-integral. If the value of 'a=b' rides arbitrarily close to 1, the value of this instance of the "law" rides arbitrarily close to 0.

This, then, is the fallacy in Evans's derivation. Although lines (2) and (4) are, we are supposing, completely true, the inference to (5)

$$\lambda x[\nabla(x=a)]b, \sim \lambda x[\nabla(x=a)]a, a=b \rightarrow (\lambda x[\nabla(x=a)]a \leftrightarrow \lambda x[\nabla(x=a)]b) \vdash a \neq b$$

can deliver a conclusion whose value is arbitrarily close to zero.

It would be nice if things could be left exactly there. Unfortunately, this treatment of Evans's derivation is open to a powerful - although, in the end, ineffective - objection. Forms of the objection have been developed by

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<sup>31</sup>Evans's intention, in the passage quoted, seems to have been to distribute  $\Delta$  across the conditional. If so, he was in error, for

$$\Delta(a=b \rightarrow (\emptyset(a) \leftrightarrow \emptyset(b))) \rightarrow (\Delta(a=b) \rightarrow \Delta(\emptyset(a) \leftrightarrow \emptyset(b)))$$

can take the value 0.

Garrett and Noonan.<sup>32</sup> (Neither Garrett nor Noonan was writing concerning fuzzy logic.) The objection is as follows.

The foregoing criticism of Evans's derivation concedes that lines (2) and (4) of the derivation are true. Moreover, these formulae are not merely true to a high degree, but determinately true, for they are derived from the assumptions ' $\nabla(a=b)$ ' and ' $\sim \nabla(a=a)$ ' respectively, and these two formulae in turn, if true, are determinately true, since in the semantics all formulae of the form ' $\nabla A$ ' are integral-valued. In other words, given only the innocuous assumption that the operation of  $\lambda$ -abstraction preserves the value of the formula to which it is applied, the assumption that  $\nabla(a=b)$  entails that it is determinately true that  $b$  has the property  $\lambda x[\nabla(x=a)]$ , and the assumption that  $\sim \nabla(a=a)$  entails that it is determinately true that  $a$  lacks that same property. So does it not further follow, just as Evans thought, that ' $a=b$ ' is determinately false? For surely the proponent of fuzzy identity means to count ' $a=b$ ' as indeterminate just in the case where it is indeterminate whether  $a$  and  $b$  have all their properties in common? If it is determinately true that  $a$  and  $b$  do *not* have all their properties in common, then how can even a fuzzy logician draw back from the conclusion that ' $a=b$ ' is determinately false?

In short, while it is true that what Evans calls Leibniz's Law is of no help in moving from (2) and (4) to (5), there is (the objection continues) a related principle that will take us directly from (2) and (4) not only to (5) but also to Evans's (5'),  $\Delta(a \neq b)$ , namely the principle that if determinately  $b$  has

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<sup>32</sup>Garrett, B. 'Vague Identity and Vague Objects' *Noûs*, 25 (1991): 341-351, pp. 347-8; Noonan, H.W. *Personal Identity* London: Routledge (1989), sect. 6.6, and 'Vague Identity Yet Again' *Analysis*, 50 (1990): 157-62, pp. 160-162.

some property that, determinately, a lacks, then, determinately, a and b are distinct. In symbols:

$$\sim \emptyset(a), \emptyset(b), \Delta(\sim \emptyset(a)), \Delta(\emptyset(b)) \vdash a \neq b \& \Delta(a \neq b).$$

(Noonan calls a weaker form of this "the principle of the Diversity of the Definitely Dissimilar".<sup>33</sup> The present principle is that of the Definite Diversity of the Definitely Dissimilar.)

This objection fails, as I shall explain.<sup>34</sup>

### *Higher-Order Indeterminacy*

A vague predicate such as 'bald' effects a tripartite division of the objects in its range of significant application, dividing them into the positive

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<sup>33</sup>'Vague Identity Yet Again', p. 160.

<sup>34</sup>In a response to Noonan, Keefe ('Contingent Identity and Vague Identity' *Analysis*, 55 (1995): 183-190) objects to Evans's derivation on the basis of a claim that delta-predicates do not denote properties. Her grounds for holding the latter are seemingly ad hoc:

If we allow that it can be genuinely indeterminate whether something has a particular property, then we must deny that 'it is indeterminate whether ... ' denotes a further property. Assuming that it does, or equivalently assuming that it can be substituted into the Diversity of the Dissimilar, begs the question against this possibility. (p. 188)

If the present treatment is correct then one can allow both that it is indeterminate whether something has a particular property P and that  $\lambda x[\nabla Px]$  is genuinely a property.

extension, those things which are definitely bald, the negative extension, those things which are definitely not bald, and the borderline cases. None of these divisions is sharp. There is no more a sharp line between the positive extension of 'bald' and the border region than there is between the positive extension and the negative extension. Russell remarked on this in 1923 (and it has been emphasised recently by several writers):

[W]ords are attributable without doubt over a certain area, but become questionable within a penumbra, outside which they are again certainly not attributable ... [T]he penumbra itself is not accurately definable, and all the vaguenesses which apply to the primary use of words apply also when we try to fix a limit to their indubitable applicability.<sup>35</sup>

As it is sometimes put, there are borderline cases of the predicate 'is a borderline case of the predicate *bald*'.<sup>36</sup> This higher-order indeterminacy or higher-order vagueness has already been mentioned in the case of the graph

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<sup>35</sup>Russell, B. 'Vagueness' *Australasian Journal of Psychology and Philosophy*, 1 (1923): 84-92, p. 87.

<sup>36</sup>The matter is discussed by, for example, Dummett, M.A.E. 'Wang's Paradox' *Synthese*, 30 (1975): 301-324; Hyde, D. 'Why Higher-Order Vagueness is a Pseudo-Problem' *Mind*, 103 (1994): 35-41; Sainsbury, R.M. 'Concepts Without Boundaries' Inaugural Lecture, 1991, King's College, London, and 'Is There Higher-Order Vagueness?' *Philosophical Quarterly*, 41 (1991): 167-182; Williamson, T. *Vagueness* London: Routledge (1994); Wright, C. 'Further Reflections on the Sorites Paradox' *Philosophical Topics*, 15 (1987): 227-90, and 'Is Higher Order Vagueness Coherent?' *Analysis*, 52 (1992): 129-39.

of  $v(\sim Bn)$  against  $n$ : the boundaries between the integral-valued sentences and the indeterminate sentences are themselves indeterminate.

A semantics for vagueness that fails to acknowledge the indeterminacy of the penumbra is at best incomplete. As Sainsbury puts it, "no sharp cut-off to the shadow of vagueness is marked in our linguistic practice, so to attribute it to [a vague] predicate is to misdescribe it".<sup>37</sup> Sainsbury believes that this presents a fundamental difficulty for fuzzy logic:

[T]his ... is what scuppers the ... descriptions of vague languages offered by fuzzy logicians ... [T]he fuzzy logician, too, whether he likes it or not, will be committed to a threefold partition: the sentences which are true to degree 1, those true to degree 0, and the remainder. But to what in our actual use of language does this division correspond? It looks as if, as before, it should correspond to the sentences true beyond the shadow of vagueness, those in some kind of borderline position, and those false beyond the shadow. But ... we do not know, cannot know, and do not need to know these supposed boundaries to use language correctly. Hence they cannot be included in a correct description of our language.  
(*ibid.*, pp. 11-12)

Fine, too, objects that fuzzy logic cannot "express the fact that Herbert [is] a *clear* borderline case of a bald man".<sup>38</sup> Yet I think fuzzy logicians will see little force in such criticisms. As the example of the graph of  $v(\sim Bn)$  against  $n$  may make clear, fuzzy logic is certainly not committed to the idea that there

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<sup>37</sup>'Concepts Without Boundaries', p. 11.

<sup>38</sup>Fine, K. 'Vagueness, Truth and Logic' *Synthese*, 30 (1975): 265-300, p. 293.

is a sharp boundary between the sentences of value 1 and the sentences of value less than 1.

What is desirable is some way of mathematicising the - as it stands - highly *unmathematical* notion of a graph with fuzzy spots. One way of proceeding is to attempt to formalise the idea, previously mentioned, that the spectrum of non-integral values is to be thought of as fuzzily partitioned into regions labelled 'nearly true', 'almost completely false', and so on. Let us take these routine English phrases seriously and let us call them and their cognates *fuzzy truth values*. As I have presented things thus far, the fuzzy truth values lie outside the formal semantics: in the mathematical theory we find only the integral and the non-integral values, in themselves perfectly sharp. The fuzzy values enter only at the level of informal talk about the mathematical theory, as in "statements with values around 0.8 or better are approximately true and those with values around 0.95 or better are almost completely true". Can the fuzzy truth values be brought within the ambit of the formal theory itself?

Zadeh has attempted to do exactly that.<sup>39</sup> Zadeh's proposal - and perhaps the best way to view it is as being in the spirit of the set-theoretic reduction of numbers to sets - is to allow fuzzy sets to do duty for fuzzy truth values. A fuzzy set is a set of pairs  $\langle x, n \rangle$ , where  $x$  is an object and  $0 < n \leq 1$ ;  $n$  represents  $x$ 's "degree of membership" in the fuzzy set.<sup>40</sup> Each fuzzy truth

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<sup>39</sup>Zadeh, L.A. 'Fuzzy Logic and Approximate Reasoning' *Synthese*, 30 (1975): 407-28; Bellman, R.E., Zadeh, L.A. 'Local and Fuzzy Logics', in Dunn, J.M., Epstein, G. (eds) *Modern Uses of Multiple-Valued Logic* Reidel: Dordrecht (1977).

<sup>40</sup>Zadeh, L.A. 'Fuzzy Sets' *Information and Control*, 8 (1965): 338-353.

value is identified with a fuzzy subset of the set consisting of the integral and non-integral values (which is to say, is identified with a fuzzy subset of the interval  $[0, 1]$ ). For example, suppose the fuzzy truth value 'approximately true' is identified with a fuzzy set  $\alpha$ . Then the degree of membership in  $\alpha$  of some non-integral value  $i$  signifies the extent to which being true to degree  $i$  counts as an instance of being approximately true. We may expect that  $\langle 0.95, 1 \rangle$  is in  $\alpha$  and that  $\forall i (\langle 0, i \rangle \notin \alpha)$ .

This set-theoretic reduction of the fuzzy truth values is not entirely satisfying. The resulting semantics hovers between pure and applied.<sup>41</sup> The problem of formalising the fuzzy truth values in a philosophically satisfying way is the principal outstanding problem of fuzzy logic, and is a challenge to philosophical logicians working on the theory of vagueness. Nevertheless, Zadeh's set-theoretic reduction of the fuzzy values amply illustrates the point that fuzzy semantics is far from committed to there being a sharp boundary between complete truth and less-than-complete truth. For two fuzzy sets can shade into one another. Points in the border region belong to both sets, although generally to different degrees. In particular, the fuzzy set 'completely true' shades into the fuzzy set 'almost completely true'. The values 1 and 0.95, for instance, will belong to both sets, although to different

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<sup>41</sup>See my 'On When a Semantics is not a Semantics' *Journal of Philosophical Logic*, 8 (1979): 399-413, 'Pure Semantics and Applied Semantics' *Topoi*, 2 (1983): 197-204, 'What is a Semantics for Classical Negation?' *Mind*, 95 (1986): 478-490, 'Vagueness and Bivalence'. *Proceedings of the Aristotelian Society*, 68 (1994): 193-200; and Lewis, D.K. 'Relevant Implication' *Theoria*, 54 (1988): 161-174, p. 173.

degrees. Fuzzy logic need impose no sharp cut-off upon the "shadow of vagueness".

I will say no more here concerning Zadeh's project of mathematicising the fuzzy truth values. The immediate problem is the semantics of the delta operators. A statement to the effect that such-and-such a statement is indeterminate may itself be indeterminate, and likewise for a statement to the effect that such-and-such a statement is determinate. Yet in the foregoing semantics, statements of the form  $\nabla A$  and  $\Delta A$  always take either the value 0 or the value 1, which is to say, are determinate. If the semantical clauses for  $\Delta$  and  $\nabla$  are to save the phenomenon of higher-order indeterminacy, they must allow statements of these forms to sometimes take non-integral values. Then the distinction between a's being a clear borderline case of a predicate F and a's being a borderline borderline case of F can be expressed in terms of the distinction between  $v(\nabla Fa)$  being 1 and  $v(\nabla Fa)$  being non-integral. Here is one way of achieving this.

With each statement A is associated A's *higher-order profile*, written  $\Omega(A)$ .  $\Omega(A)$  is a subset of  $[0, 1] \times [0, 1]$ , which is to say, is a set of ordered pairs, each member of each pair being either 1, 0, or non-integral. The idea is that each member  $\langle n, i \rangle$  of  $\Omega(A)$  records the degree i to which it is true that A possesses the value n. I will call the right-hand members of pairs in a profile *higher-order degrees* and the left-hand members *primary degrees*. (All the remarks made earlier concerning "excess precision", idealisation, ranking, and so forth, apply equally to these new higher-order degrees.) Where A definitely has the value n, then  $\langle n, 1 \rangle \in \Omega(A)$ .  $\Omega(A)$  respects the conditions:

1.  $\forall i \forall j \forall k (\langle i, j \rangle \in \Omega(A) \rightarrow (\langle i, k \rangle \in \Omega(A) \rightarrow j=k))$
2.  $\forall i \forall j \forall k (\langle i, 1 \rangle \in \Omega(A) \rightarrow (\langle j, k \rangle \in \Omega(A) \rightarrow \langle j, k \rangle = \langle i, 1 \rangle))$ .

If there is an  $n$  such that  $\langle n, 1 \rangle \in \Omega(A)$ ,  $\Omega(A)$  is said to be *unitary*. Where it is indeterminate whether  $A$  has a certain value  $n$ , this is marked by the presence in the higher-order profile of a pair  $\langle n, i \rangle$  where  $i$  is non-integral. The profile may contain a number of such pairs, each containing a different primary value.

The "fuzzy spots" or regions of higher-order indeterminacy on the graph of  $v(\sim Bn)$  against  $n$  are regions where  $\Omega(\sim Bn)$  is not unitary. For example, let us pick numbers  $x, y$  and  $z$  ( $x < y < z$ ) such that  $x$  and  $y$  lie in a fuzzy region and  $z$  lies clearly outside it on a crisp line.  $\Omega(\sim Bx)$  is, say,  $\{\langle 1, 0.8 \rangle, \langle 0.9, 0.9 \rangle\}$ ,  $\Omega(\sim By)$  is  $\{\langle 1, 0.95 \rangle\}$  and  $\Omega(\sim Bz)$  is  $\{\langle 1, 1 \rangle\}$ . At  $n=z$  the line is definite. At  $n=y$  it is not so definite, drawn fainter perhaps, while at  $n=x$  the line has disappeared altogether.

There are three cases to be considered in reformulating the semantical clauses for  $\Delta$  and  $\nabla$ .

Case 1  $\Omega(A)$  is unitary. The member of  $\Omega(A)$  is either  $\langle 1, 1 \rangle$  or  $\langle 0, 1 \rangle$  or  $\langle i, 1 \rangle$  for some non-integral  $i$ . In other words,  $A$  is either determinately determinate or determinately indeterminate. As before, where the primary degree is integral,  $v(\Delta A)=1$  and  $v(\nabla A)=0$ , and where the primary degree is non-integral,  $v(\nabla A)=1$  and  $v(\Delta A)=0$ .

Case 2  $\Omega(A)$  is non-unitary and none of the primary degrees in  $\Omega(A)$  is integral. Here it is indeterminate which primary degree  $A$  has, but it is not indeterminate whether or not  $A$  is indeterminate.  $v(\nabla A)=1$  and  $v(\Delta A)=0$ .

Case 3  $\Omega(A)$  is non-unitary and at least one of the primary degrees in  $\Omega(A)$  is integral. Here it is indeterminate whether  $A$  is determinate. Either  $\langle 1, i \rangle \in \Omega(A)$  or  $\langle 0, j \rangle \in \Omega(A)$  or both are, for some non-integral  $i$  and  $j$ . If only the first of these pairs is present then  $v(\Delta A)=i$ , and if only the second is present  $v(\Delta A)=j$ . If both are present,  $v(\Delta A)=\max(i, j)$ .  $v(\nabla A)$  is  $1 - v(\Delta A)$

(thus, for example, if it is almost completely true that A is determinate, then it is almost completely false that A is indeterminate). Assuming that  $v(\Delta \sim A) = v(\Delta A)$ , this treatment of  $\nabla$  preserves duality.

Each  $\Omega(A)$  considered so far consists of ordered pairs. Higher-order profiles of this sort are called *two-dimensional*. As promised, the use of 2-D higher-order profiles fuzzifies the division between the determinate and the indeterminate. Still higher-order divisions remain unfuzzy. For example, the two-dimensional apparatus leaves sharp the division between those sentences that are determinately indeterminate and those that are indeterminately indeterminate. This division too can be fuzzified if desired, at the price of adding one further dimension to each higher-order profile. A 3-D higher-order profile consists of triples  $\langle n, i, j \rangle$ .  $j$  is the degree to which it is true that  $i$  is the degree to which it is true that the sentence in question possesses the value  $n$ . This process can be continued indefinitely. Adding a fourth dimension fuzzifies third-order divisions, such as that between the determinately indeterminately indeterminate and the indeterminately indeterminately indeterminate. Where no such boundary is to remain sharp the members of the higher-order profiles are infinite vectors  $\langle x_1, x_2, \dots, x_n, \dots \rangle$  (and orders of indeterminateness that are even higher still can be coded into a profile, if desired, by employing vectors of increasing order-type). It is, of course, only the finite cases that are of use to engineers, and how low a dimensionality one can get away with will depend on the software project in hand. There is no point in paying for dimensions that serve only to fuzzify boundaries too fine to be of any practical relevance.

Let us return to the discussion of Evans's argument and recall that the crucial step in the objection from the principle of the Definite Diversity of the Definitely Dissimilar, above, was the claim that the proof gives us a property,

$\lambda x[\nabla(x=a)]$ , which, determinately, b has and a does not have. As Noonan puts it:

What the Evans ... argument demonstrates is that indefinite identity, so conceived, can have no application; in the case in which 'a=b' is indeterminate in truth-value ... there will be a predicate 'it is indeterminate whether x=a' definitely true of b, whose negation is definitely true of a.<sup>42</sup>

Noonan accuses critics of Evans's argument of rejecting the principle that if "there is a predicate definitely true of b whose negation is definitely true of a [then it is not] indeterminate whether a is b", saying that the rejection of this principle is "the *only* road forward for an opponent of Evans's position" (Noonan's italics).<sup>43</sup>

The fallacy is plain. Once higher-order indeterminacy is acknowledged one cannot rely on an indeterminate statement being determinately indeterminate. (So here is a second reason why Evans was mistaken in his claim that  $\Delta$  and  $\nabla$  "generate a modal logic as strong as S5":  $\nabla A$  does not imply  $\Delta \nabla A$ .<sup>44</sup>) That it is indeterminate whether a=b does *not* entail that there is "a

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<sup>42</sup>*Personal Identity*, pp. 135-6. For reasons of uniformity I have replaced 'it is indeterminate whether x=b', which occurs in the original, by 'it is indeterminate whether x=a', and I have replaced the immediately following occurrence of 'a' by 'b' and the immediately following occurrence of 'b' by 'a'. I have made similar replacements in the next quotation.

<sup>43</sup>'Vague Identity Yet Again', pp. 160-162.

<sup>44</sup>(See also Garrett, B. 'Vagueness, Identity, and the World' *Logique et Analyse*, 135-136 (1991): 349-358, p. 354.) As Dummett pointed out as early as 1975, the logic will be weaker than S4, since  $\Delta A$  does not imply  $\Delta \Delta A$

predicate ... definitely true of b, whose negation is definitely true of a". The statement ' $\nabla(a=b)$ ' does entail that  $\lambda x[\nabla(x=a)]b$ , but both these statements may themselves be indeterminate. Critics of Evans's argument certainly need not reject the principle that Noonan states. In my view, the principle of the Definite Diversity of the Definitely Dissimilar is self-evidently true (whatever that means), but it is of no more help than Leibniz's Law in the attempt to justify the move from (2) and (4) to (5) and (5').

If Evans's derivation is enriched with the further assumption that ' $\nabla(a=b)$ ' is determinate, and the principle of the Definite Diversity of the Definitely Dissimilar is employed, then the derivation of (5) and (5') may seem to go through successfully (although for some doubts of a different nature see my "On Vague Objects, Fuzzy Logic and Fractal Boundaries", *op. cit.*). Discharging the new assumption by reductio ad absurdum produces a derivation of

$$\nabla(a=b) \vdash \sim\Delta\nabla(a=b).$$

If the derivation holds good, this is an interesting and important result (and perhaps Evans would have been pleased enough to emerge as discoverer of a new property of vague identity<sup>45</sup>). Employing the usual metaphors, the result tells us that if a predicate ' $x=a$ ' is vague then the 'shadow of vagueness' that it

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('Wang's Paradox', p. 311). As I have already mentioned, neither the T-axiom  $\Box A \rightarrow A$  nor the K-axiom  $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$  holds when ' $\Box$ ' is replaced by ' $\Delta$ '.

<sup>45</sup> See Burgess, J.A. 'Vague Identity: Evans Misrepresented' *Analysis*, 49 (1989): 112-119.

casts is *all* penumbra: all the borderline cases are borderline borderline cases.<sup>46</sup>

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<sup>46</sup>The strain on the traditional metaphor of a shadow is evident. Perhaps a better metaphor is provided by fractal surfaces such as the Sierpinski carpet (Mandelbrot, B.B. *The Fractal Geometry of Nature* New York: Freeman (1977, revised and expanded 1982, 1983)). This is a two-dimensional surface which is "all edge": like a shadow, the Sierpinski carpet covers a finite area, but every point on the carpet lies on an edge of the carpet. An interesting route of enquiry is to fuzzify the Sierpinski carpet.